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Comparison and aggregation of max-plus linear systems

James Ledoux* and Laurent Truffet†

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Abstract

We study linear systems in the max-plus algebra, where the basic operations are maximum and addition. We define a preorder to compare the state vectors of max-plus linear systems with the same dimension. We provide two algebraic methods to get bounds (with respect to this preorder) on the state vectors of a lumped max-plus linear system. The first method is based on the strong lumpability. The second method is based on the coherency property, which also allows one to provide bounds on the state vectors of the original linear system from those for the lumped system. We provide the algorithms to compute all the proposed bounds. We show that they can be used for models with a large state index set by means of a time and space complexity analysis.

KEYWORD: lumpability
2000 MSC: 15A45; 16Y60; 39B72; 93C65

1 Introduction

A finite dimensional dynamical system is said to be linear if its state vectors $x(n)$ $n \geq 1$, are given by the following autonomous difference (or state) equation

$$x(0) \in \mathbb{R}^{\eta \times 1},$$
$$x(n+1) = A\,x(n) \Longleftrightarrow x_i(n+1) = \sum_{j=1}^{\eta} a_{i,j}\,x_j(n) \quad i = 1,\dots,\eta \tag{1}$$

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for some matrix $A = [a_{i,j}] \in \mathbb{R}^{\eta\times\eta}$. In this paper, we consider the counterpart of such a description of a dynamical system when we replace the set $\mathbb{R}$ by $\mathbb{R}_{\max} \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$ and the usual operations $(+, \times)$ by the operations denoted by $(\oplus, \otimes)$:

$$a \oplus b \stackrel{\text{def}}{=} \max(a, b) \qquad a \otimes b \stackrel{\text{def}}{=} a + b \quad a, b \in \mathbb{R}_{\max}.$$

A max-plus linear system is a system where the state vector $x(n)$ satisfies an equation as Equation (1) with the new operations $(\oplus, \otimes)$. Max-plus linear systems cover a large variety of problems occurring when analyzing the behavior of discrete event systems [1], [3], [4], [2]. Let us consider a naive example to give some insight into the different concepts introduced here. We have an activity network represented by the weighted directed graph in Figure 1. Entry $a_{i,j}$ corresponds to the arc from node j to node i. This arc can be

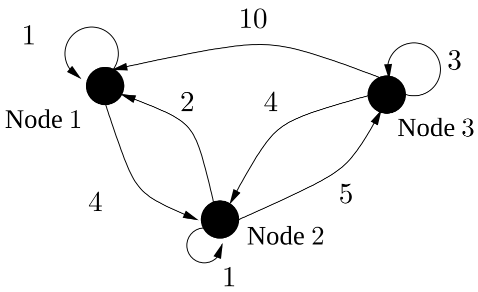

Figure 1: Activity network

interpreted as an output channel for node j, and simultaneously, as an input channel for node i. Suppose that the node i starts its activity as soon as all preceding nodes have sent their results to node i. Then, the following equation

$$n \geq 0: \qquad i = 1, 2, 3 \quad x_i(n+1) = \max_{j=1,2,3} \big(a_{i,j} + x_j(n)\big) \tag{2}$$

describes when activities take place. The interpretation of the quantities involved in the above equation is:

- $x_i(n)$ is the earliest epoch at which node i becomes active for the nth time;
- $a_{i,j}$ is the sum of the activity time of node j and the traveling time from node j to node i.

The fact that we write $a_{i,j}$ for a quantity connected to the arc from node j to node i has to do with matrix equations which will be written with column vectors.

The core of this paper is the *comparison of the dynamics* of such max-plus linear systems. Usually, the comparison between two state vectors is made component by component [3]. We will introduce a preorder on $\mathbb{R}^{\eta}_{\max}$, denoted by $\leq_{\mathbb{K}}$, and defined by

$$x, y \in \mathbb{R}^{\eta \times 1}, \qquad x \leq_{\mathbb{K}} y \quad \text{iff} \quad \oplus_{j=i}^{\eta} x_j \leq \oplus_{j=i}^{\eta} y_j \quad i = 1, \ldots, \eta. \tag{3}$$

It is clear that the preorder $\leq_{\mathbb{K}}$ is weaker than the component-wise preorder. Indeed, if the vectors x and y are such that $x \leq y$ component-wise, then $x \leq_{\mathbb{K}} y$. The converse is false in general. We can also compare two matrices A and B, with $A \leq_{\mathbb{K}} B$ if Inequality (3) holds column by column. The preorder $\leq_{\mathbb{K}}$ is the analogue of the strong stochastic order for non-negative vectors/matrices [5]. Comparison between two dynamics with respect to the preorder $\leq_{\mathbb{K}}$, means that we are interested in inequalities as

$$n \geq 1, \quad x^1(n) \leq_{\mathbb{K}} x^2(n)$$

where $\{x^1(n), n \geq 1\}$ and $\{x^2(n), n \geq 1\}$ are the state vectors associated with two linear max-plus systems. Let us turn back to our example. Consider the two different initial data $x^1(0)$ and $x^2(0)$. We get two families of state vectors $\{x^1(n), n \geq 1\}$ and $\{x^2(n), \geq 1\}$ from the difference equation (2). Then, we have $x^1(n) \leq_{\mathbb{K}} x^2(n)$ if and only if

> for every $i = 1, 2, 3$, the earliest epoch at which the nodes $i, \ldots, 3$ have all become active for the nth time for the first dynamics is less than the corresponding quantity for the second dynamics.

We will define the concept of monotonicity for a matrix with respect to $\leq_{\mathbb{K}}$ (see [5] for a stochastic matrix). In fact, dealing with a $\mathbb{K}$-monotone matrix A ensures that any $\mathbb{K}$-inequality between two vectors is preserved by $\otimes$-multiplication to the left by matrix A

$$x \leq_{\mathbb{K}} y \Longrightarrow A \otimes x \leq_{\mathbb{K}} A \otimes y.$$

Firstly, we will show that any square matrix A is bounded from above (resp. below) by a $\mathbb{K}$-monotone matrix U (resp. L). These bounds are optimal in a sense to be specified later. The main interest in these results is to assert that we can always $\mathbb{K}$-majorize the state vectors of a linear max-plus system through the construction of $\mathbb{K}$-monotone bounds of the matrix governing the linear system. Indeed, if the initial data are such that $l(0) \leq_{\mathbb{K}} x(0) \leq_{\mathbb{K}} u(0)$, then

$$l(n) \stackrel{\text{def}}{=} L^{\otimes n} \otimes l(0) \ \leq_{\mathbb{K}} \ x(n) \ \leq_{\mathbb{K}} \ u(n) \stackrel{\text{def}}{=} U^{\otimes n} \otimes u(0) \quad n \geq 1.$$

Secondly, we consider the dynamics of a lumped system. Indeed, let us define a surjective map ϕ from the state index set, say $S = \{1, \ldots, \eta\}$, of the linear system into the set $\Sigma = \{1, \ldots, N\}$ with $1 \leq N < \eta$. Such a map will called be a *lumping map*. We assume that ϕ is non-decreasing for notational convenience. We associate with the map ϕ a lumping matrix $V \in \mathbb{R}_{\max}^{N \times \eta}$ defined by

$$\forall I \in \Sigma, \forall j \in S \quad v_{I,j} = \delta_{\{\phi(j)=I\}}, \tag{4}$$

where the $\{-\infty, 0\}$-valued function $\delta_{\{\cdot\}}$ is 0 if the logical assertion $\{\cdot\}$ is true, and $-\infty$ otherwise. Then, we deal with the following system of state equations

$$\left.\begin{array}{lrl} & x(0) & \in \mathbb{R}_{\max}^{\eta \times 1} \\ \text{(I)} & x(n+1) & = A \otimes x(n) \\ \text{(II)} & y(n) & = V \otimes x(n) \end{array}\right\} \tag{5}$$

where $A \in \mathbb{R}_{\max}^{\eta \times \eta}$. In general, the vectors $\{y(n), n \geq 1\}$ do not verify a difference equation as Equation (5,(I)). A condition under which there exists some matrix $\widehat{A} \in \mathbb{R}_{\max}^{N \times N}$ such that

$$y(n+1) = \widehat{A} \otimes y(n) \quad n \geq 1,$$

is called a *lumpability condition* [8]. These lumpability conditions are the counterparts of those existing for Markov chains [6]. For our activity network, considering the lumping map ϕ_1 from $\{1, 2, 3\}$ into $\{1, 2\}$ defined by $\phi_1(1) = \phi_1(2) = 1, \phi_1(3) = 2$, means that the behavior of the system is observed through the couple of values $y_1(n) = \max\big(x_1(n), x_2(n)\big)$ and $y_2(n) = x_3(n)$. In other words, the output of the system is only the earliest epoch at which the nodes 1 and 2 (resp. node 3) are active for the nth time. Roughly speaking, the activity network in Figure 1 will be lumpable with respect to ϕ_1 if the vectors $y(n)$, $n \geq 1$, satisfy a difference equation. Therefore, the network with three nodes can be replaced by a 2-nodes network (lumping nodes 1 and 2) without loss of the linear characteristic of the corresponding dynamical system.

Thirdly, our goal is still to compute $\mathbb{K}$-bounds on the aggregated state vector $y(n)$ ($n \geq 1$) defined by Equation (5,(II)). This kind of issue arises when the state index set S is (very) large and

1. we can only consider the dynamics of an aggregated system from the computational point of view;

2. or we are only interested in assessing the state vector $y(n)$ of the system. For instance, when concerned with the computation of a performance or cost measure which only depends on the state vector $y(n)$. In the aforementioned network, one could consider scheduling a monitoring task of the simultaneous activity of nodes 1 and 2.

The proposed bounds come from combining

- the construction of monotone bounds of the matrix governing the dynamics of the system as described in the first step
- and the use of lumpability conditions.

The results are as follows. For each selected lumpability condition, we show that for any matrix A and any lumping map ϕ, there always exist $\mathbb{K}$-bounds L and U of A that are lumpable with respect to ϕ. Additionally, if $l(0) \leq_{\mathbb{K}} y(0) \leq_{\mathbb{K}} u(0)$, then we will have

$$\widehat{l}(n) \stackrel{\text{def}}{=} \widehat{L}^{\otimes n} \otimes \widehat{l}(0) \ \leq_{\mathbb{K}} \ y(n) \ \leq_{\mathbb{K}} \ \widehat{u}(n) \stackrel{\text{def}}{=} \widehat{U}^{\otimes n} \otimes \widehat{u}(0), \quad n \geq 1$$

for some $N \times N$-matrices $\widehat{L}$ and $\widehat{U}$ (where $\widehat{l}(0) = V \otimes l(0)$ and $\widehat{u}(0) = V \otimes u(0)$). We mainly use the so-called coherency property (see [8] and references cited therein). It also allows one to derive $\mathbb{K}$-bounds on the original state vector $x(n)$ from computation with the lumped linear system.

Each existence theorem provided in this paper is supported by a constructive proof. This allows one to develop algorithms. Their complexity shows that they are efficient when the state index set S is large.

The paper is organized as follows. In Section 2, we report the main notation of the paper while introducing the framework of linear (dynamical) systems in the max-plus algebra. In Section 3, we present the results for the comparison of the state vectors of systems with the same state space. These results are based on a pioneering paper [9]. In Section 4, we provide the methods to compute monotone bounds on a given matrix. In Section 5, we provide the methodology for bounding the state vectors of aggregated systems. All results will be illustrated by a simple example. In Section 6, we give the algorithms to compute the various bounds. Their complexity is analyzed. We conclude in Section 7.

2 Notation and definitions

In this Section we follow Baccelli et al. [1, Chap 3] excepting some notation changes which are motivated by the setting of this paper.

2.1 Max-plus algebra

$(\mathbb{R}_{\max}, \oplus, \otimes)$ has a zero denoted by $\mathbb{0}$ (here $\mathbb{0} = -\infty$) and an unit element denoted by $\mathbb{1}$ (here $\mathbb{1} = 0$) [1]. The law $\oplus$ is idempotent, i.e. $a \oplus a = a$ for any $a \in \mathbb{R}_{\max}$. The element $\mathbb{0}$ is absorbing for $\otimes$. "Max-plus algebra" is the common name of the idempotent semiring $(\mathbb{R}_{\max}, \oplus, \otimes)$.

The usual order relation on $\mathbb{R}_{\max}$ can be defined using $\oplus$ by:

$$a, b \in \mathbb{R}_{\max}, \quad (a \leq b \Longleftrightarrow a \oplus b = b).$$

In this paper, the inverse of any real a w.r.t. the $\otimes$-operation is denoted by $-a$ (let us note that we do not use the one or two-dimensional display notation of [1, p105]). Thus, $b - a$ stands for $b \otimes (-a)$. Note that $\mathbb{0} - a = \mathbb{0}$ for any $a \in \mathbb{R}$.

The vectors are column-vectors except special mention. $(\cdot)^\top$ denotes the transpose operator.

$\mathbb{1}_n$ (resp. $\mathbb{0}_n$) denotes the n-dimensional column-vector having all components equal to $\mathbb{1}$ (resp. $\mathbb{0}$).

We recall that the $\{\mathbb{0}, \mathbb{1}\}$-valued function $\delta_{\{\cdot\}}$ is $\mathbb{1}$ if logical assertion $\{\cdot\}$ is true and $\mathbb{0}$ otherwise.

For any matrix $A = [a_{i,j}] \in \mathbb{R}_{\max}^{n \times p}$, $a_{i,\cdot}$ and $a_{\cdot,j}$ denote its ith row and jth column respectively. To avoid a heavy use of the transpose operator in the formulae, $a_{i,\cdot}$ will be considered as a row-vector, i.e. $a_{i,\cdot} \in \mathbb{R}_{\max}^{1 \times p}$. We need define operations on the matrices with entries in $\mathbb{R}_{\max}$. Let us define the external multiplication by

$$\lambda \in \mathbb{R}_{\max}, A = [a_{i,j}] \in \mathbb{R}_{\max}^{n \times p}, \qquad \lambda \otimes A \stackrel{\text{def}}{=} [\lambda \otimes a_{i,j} = \lambda + a_{i,j}]_{i=1,\ldots,n; j=1,\ldots,p}$$

If $A \in \mathbb{R}_{\max}^{n \times p}$ and $B \in \mathbb{R}_{\max}^{p \times q}$, the product $A \otimes B$ is defined by

$$A \otimes B \stackrel{\text{def}}{=} \left[\bigoplus_{k=1}^{p} a_{i,k} \otimes b_{k,j} = \max_{k=1,\ldots,p} (a_{i,k} + b_{k,j}) \right]_{i=1,\ldots,n; j=1,\ldots,q}.$$

The sum $A \oplus B$ of two matrices $A \in \mathbb{R}_{\max}^{n \times p}$ and $B \in \mathbb{R}_{\max}^{n \times p}$, is defined by

$$A \oplus B \stackrel{\text{def}}{=} \left[a_{i,j} \oplus b_{i,j} = \max(a_{i,j} \, , \, b_{i,j}) \right]_{i=1,\ldots n; j=1,\ldots,p}$$

[1] We use this notation to do the parallel with results in the usual algebra. In [1], $\mathbb{0}$ (resp. $\mathbb{1}$) is denoted by ϵ (resp. e).

2.2 Autonomous dynamics and aggregated dynamics

Let us consider a lumping map ϕ from $S = \{1, \dots, \eta\}$ into $\Sigma = \{1, \dots, N\}$ with $1 \leq N < \eta$. Matrix V is the corresponding lumping matrix defined by Equation (4). In this paper, we study systems for which the dynamical behavior is determined by System (5) of autonomous difference (or state) equations. The series $< x(n) >_{n=0}^{+\infty}$ defined by Equation (5, (I)) will be called an autonomous (linear) dynamics. It is specified by the 2-tuple $(x(0), A)$. The series $< y(n) >_{n=0}^{+\infty}$ defined by Equation (5,(I),(II)), will be called the aggregated dynamics.

3 Comparison of the state vectors of linear systems with the same state space

The aim of this section is to present some results for comparing (w.r.t. the $\leq_{\mathbb{K}}$ preorder) the two autonomous dynamics $(z(0), A)$ and $(t(0), B)$ with $z(0), t(0) \in \mathbb{R}_{\max}^{\eta\times 1}$ and $A, B \in \mathbb{R}_{\max}^{\eta\times\eta}$. They are based on the property of $\mathbb{K}$-monotonicity of a matrix, which ensures that any $\mathbb{K}$-inequality between two vectors, will be preserved under the multiplication to the left by the matrix. The main result (Theorem 3.2) gives a condition under which the two dynamics $(z(0), A)$ and $(t(0), B)$ may be compared. This section is a slight extension of the work in the pioneering paper [9] dealing with Bellman-Maslov chains. All statements are inspired by results on monotone Markov chains [5].

Definition 3.1 ($\mathbb{K}_n$-comparison) *Let x, y be two elements of $\mathbb{R}_{\max}^{n\times 1}$. We say that x is $\mathbb{K}_n$-smaller than y iff*

$$\mathbb{K}_n \otimes x \leq \mathbb{K}_n \otimes y \text{ (component-wise)}, \tag{6}$$

where $\mathbb{K}_n$ is the $(n \times n)$-dimensional matrix defined by

$$\mathbb{K}_n \stackrel{\text{def}}{=} [\delta_{\{i\leq j\}}]_{1\leq i,j\leq n} = \begin{pmatrix} \mathbb{1} & \cdots & \cdots & \mathbb{1} \\ \mathbb{0} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbb{0} & \cdots & \mathbb{0} & \mathbb{1} \end{pmatrix}. \tag{7}$$

If Condition (6) *is fulfilled, then we write $x \leq_{\mathbb{K}_n} y$. Sometimes, the dimensional argument, i.e. n, will be omitted.*

The $\mathbb{K}_n$-comparison of two matrices $A, B \in \mathbb{R}_{\max}^{p\times n}$ is naturally defined by

$$\begin{aligned} A \leq_{\mathbb{K}_n} B \quad &\iff \quad \mathbb{K}_n \otimes A \leq \mathbb{K}_n \otimes B \qquad \text{(coefficient-wise)} \\ &\iff \quad a_{\cdot,j} \leq_{\mathbb{K}_n} b_{\cdot,j} \quad j = 1, \dots, n. \end{aligned}$$

It is easily seen that relation $\leq_{\mathbb{K}_n}$ is reflexive and transitive on $\mathbb{R}^n_{\max}$, that is, $\leq_{\mathbb{K}_n}$ defines a preorder on $\mathbb{R}^n_{\max}$.

Another important concept for comparison is monotonicity, which is defined as follows.

Definition 3.2 ($\mathbb{K}$-monotone matrix) *Let A be an element of $\mathbb{R}^{n\times n}_{\max}$. Matrix A is said to be $\mathbb{K}_n$-monotone iff*

$$\forall x, y \in \mathbb{R}^{n\times 1}_{\max}, \quad \big((x \leq_{\mathbb{K}_n} y) \Longrightarrow (A \otimes x \leq_{\mathbb{K}_n} A \otimes y) \big). \tag{8}$$

The next theorem provides a tractable criterion for $\mathbb{K}$-monotonicity.

Theorem 3.1 (Criterion for $\mathbb{K}$-monotonicity) *Let A be an element of $\mathbb{R}^{n\times n}_{\max}$. A is said to be $\mathbb{K}_n$-monotone iff*

$$j = 1, \ldots, n-1, \quad a_{\cdot,j} \leq_{\mathbb{K}_n} a_{\cdot,j+1}, \tag{9}$$

recalling that $a_{\cdot,j}$ denotes the jth column of A.

Proof. *(Only If).* Let us note that $e(j) \leq_{\mathbb{K}_n} e(j+1)$, $j = 1, \ldots n-1$, if $e(j)$ denotes the n-dimensional vector where the jth component is $\mathbb{1}$ and the others are $\mathbb{0}$. Thus, $A \otimes e(j) = a_{\cdot,j} \leq_{\mathbb{K}_n} A \otimes e(j+1) = a_{\cdot,j+1}$ since A is $\mathbb{K}$-monotone.

(If). Let us consider $x, y \in \mathbb{R}^{n\times 1}_{\max}$ such that $x \leq_{\mathbb{K}_n} y$. We write

$$\mathbb{K}_n \otimes A \otimes y = \oplus_{j=1}^n \mathbb{K}_n \otimes a_{\cdot,j} \otimes y_j. \tag{10}$$

It follows from (9) and the transitivity of $\leq$ that

$$\mathbb{K}_n \otimes a_{\cdot,1} \leq \mathbb{K}_n \otimes a_{\cdot,2} \leq \cdots \leq \mathbb{K}_n \otimes a_{\cdot,n}.$$

This could be rewritten using idempotency of $\oplus$

$$j = 2, \ldots, n \quad \mathbb{K}_n \otimes a_{\cdot,j} = \bigoplus_{k=1}^{j} \mathbb{K}_n \otimes a_{\cdot,k} \tag{11}$$

Using Equation (11), the associativity of $\oplus$ and the distributivity of $\oplus$ over $\otimes$, we get

$$\mathbb{K}_n \otimes A \otimes y = \bigoplus_{k=1}^{n} \mathbb{K}_n \otimes a_{\cdot,k} \otimes (\oplus_{j=k}^n y_j)$$

Since $x \leq_{\mathbb{K}} y$, i.e. for every k, $(\oplus_{j=k}^n x_j) \oplus (\oplus_{j=k}^n y_j) = \oplus_{j=k}^n y_j$, we obtain

$$\mathbb{K}_n \otimes A \otimes y = \mathbb{K}_n \otimes A \otimes x \oplus \mathbb{K}_n \otimes A \otimes y \text{ (component-wise)},$$

or $A \otimes x \leq_{\mathbb{K}_n} A \otimes y$. ■

We state now the main result of this section. It is an extension of [9, Th 3.2].

Theorem 3.2 ($\mathbb{K}$-comparison of autonomous dynamics) *Let $(z(0), A)$ and $(t(0), B)$ be two η-dimensional autonomous dynamics. If the following conditions hold*

(i) $z(0) \leq_{\mathbb{K}_\eta} t(0)$,

(ii) $A \leq_{\mathbb{K}_\eta} B$

(iii) *A or B is $\mathbb{K}_\eta$-monotone*

then

$$\forall n \geq 0, \quad z(n) = A^{\otimes n} \otimes z(0) \leq_{\mathbb{K}_\eta} t(n) = B^{\otimes n} \otimes t(0).$$

Proof. Suppose that A is $\mathbb{K}_\eta$-monotone. We have from Inequality (ii)

$$\mathbb{K}_\eta \otimes A \otimes t(0) \leq \mathbb{K}_\eta \otimes B \otimes t(0).$$

Since Inequality (i) holds, we can apply Relation (8) to $x = z(0)$, $y = t(0)$ and the matrix A. We get

$$\mathbb{K}_\eta \otimes A \otimes z(0) \leq \mathbb{K}_\eta \otimes A \otimes t(0).$$

By the transitivity of $\leq$, we obtain

$$\mathbb{K}_\eta \otimes A \otimes z(0) \leq \mathbb{K}_\eta \otimes B \otimes t(0).$$

Thus, we prove that, if $z(0) \leq_{\mathbb{K}_\eta} t(0)$, then $z(1) \leq_{\mathbb{K}_\eta} t(1)$. Now, the proof is easily completed by induction on n. ■

4 Construction of a $\mathbb{K}$-monotone bound

We assume in Theorem 3.2 that at least one of the two autonomous dynamics is governed by a monotone matrix, but it does not always hold. However, it will follow from Theorems 4.1 and 4.3 that the matrix governing any given autonomous dynamics is bounded from above and from below by a $\mathbb{K}$-monotone matrix. Specifically, for any squared matrix A, there exists $\mathbb{K}$-monotone matrices A^- and A^+ such that

$$A^- \leq_{\mathbb{K}} A \leq_{\mathbb{K}} A^+.$$

Hence, Theorem 3.2 ensures that, if $l(0) \leq_{\mathbb{K}} x(0) \leq_{\mathbb{K}} u(0)$, then

$$l(n) = A^{-\otimes n} \otimes l(0) \ \leq_{\mathbb{K}} x(n) = A^{\otimes n} \otimes x(0) \ \leq_{\mathbb{K}} \ u(n) = A^{+\otimes n} \otimes u(0) \quad n \geq 1.$$

The $\mathbb{K}$-bounds A^- and A^+ are also shown to be optimal w.r.t. preorder $\leq_{\mathbb{K}}$.

4.1 Upper bound

Given a matrix $A \in \mathbb{R}_{\max}^{n \times n}$, we show in Theorem 4.1 that there always exists a $\mathbb{K}$-monotone matrix A^+ such that

$$\begin{array}{ll} (a) & A \leq_{\mathbb{K}} A^+ \\ (b) & \text{for any monotone } C \text{ such that } A \leq_{\mathbb{K}} C, \text{ we have } A^+ \leq_{\mathbb{K}} C. \end{array}$$

So, A^+ is said to be a monotone upper bound on A w.r.t. the preorder $\leq_{\mathbb{K}}$ Construction of such a matrix A^+ is based on the following lemma.

Lemma 4.1 *Let a, b, c be three elements of $\mathbb{R}_{\max}$. Let us consider the system of inequalities $\mathcal{U}\big(a, b, c\big)$ defined by*

$$\left\{ \begin{array}{l} a \oplus b \leq x \oplus c \\ b \leq c. \end{array} \right. \tag{12}$$

The solution set of system $\mathcal{U}\big(a, b, c\big)$ over $\mathbb{R}_{\max}$ is $[x^-(a, b, c), +\infty[$ where $x^-(a, b, c) = a \otimes \delta_{\{c<a\}}$.

Proof. It is easily checked that $x^-(a, b, c)$ is a solution of Inequalities (12). Let us show that $x^-(a, b, c)$ is the smallest solution. Let y be another solution of Inequalities (12). If $c < a$ then we have $x^-(a, b, c) = a = a \oplus b \leq y \oplus c$. This implies that $x^-(a, b, c) \leq y$. The case $c \geq a$ is obvious, since $\mathbb{0}$ is the minimal element of $\mathbb{R}_{\max}$. Since $\max(\cdot, c)$ is a non-decreasing function, it is clear that any $x \geq x^-(a, b, c)$ is also a solution of Inequalities (12). ■

Now, we state the main result of this subsection.

Theorem 4.1 (optimal $\mathbb{K}$-monotone upper bound) *Let A be an element of $\mathbb{R}_{\max}^{\eta \times \eta}$. Then, there exists a matrix $A^+ \in \mathbb{R}_{\max}^{\eta \times \eta}$ such that*

$$\left. \begin{array}{ll} (a) & A \leq_{\mathbb{K}} A^+ \\ (b) & A^+ \textit{ is } \mathbb{K}\textit{-monotone} \\ (c) & \textit{for any monotone } C \textit{ such that } A \leq_{\mathbb{K}} C, \textit{ we have } A^+ \leq_{\mathbb{K}} C. \end{array} \right\} \tag{13}$$

Proof. System (13) may be rewritten as (see Theorem 3.1 for (b))

$$\begin{array}{ll} (a) & \text{for } j = 1, \ldots, \eta, \quad \mathbb{K} \otimes a_{\cdot,j} \leq \mathbb{K} \otimes a^+_{\cdot,j} \\ (b) & \text{for } j = 2, \ldots, \eta \quad \mathbb{K} \otimes a^+_{\cdot,j-1} \leq \mathbb{K} \otimes a^+_{\cdot,j} \\ (c) & \text{for any monotone } C \text{ verifying } A \leq_{\mathbb{K}} C, \text{ we have } \mathbb{K} \otimes a^+_{\cdot,j} \leq \mathbb{K} \otimes c_{\cdot,j}, \\ & j = 1, \ldots, \eta. \end{array}$$

The construction of A^+ is by induction on the column number $j \in S$.
First, we set $a^+_{\cdot,1} = a_{\cdot,1}$.
Assume now the construction of $a^+_{\cdot,k}$, $k = 1, \ldots, j-1$ with $j > 1$ to be done. The jth row of A^+, $a^+_{\cdot,j}$, will be defined by a backward induction on the component number i. With convention that $\mathbb{k}_{\eta+1,\cdot} = \mathbb{0}_\eta$, we have to solve

$$\mathcal{U}\big(a_{i,j}\,,\ \mathbb{k}_{i+1,\cdot} \otimes a_{\cdot,j}\,,\ \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}\big) \quad \text{and} \quad \mathcal{U}\big(a^+_{i,j-1}\,,\ \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j-1}\,,\ \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}\big).$$

From Lemma 4.1, a minimal solution is given by

$$a^+_{i,j} = x^-\big(a_{i,j}\,,\ \mathbb{k}_{i+1,\cdot} \otimes a_{\cdot,j}\,,\ \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}\big) \oplus x^-\big(a^+_{i,j-1}\,,\ \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j-1}\,,\ \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}\big).$$

Or, equivalently

$$a^+_{i,j} = a_{i,j} \otimes \delta_{\{a_{i,j} > \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}\}} \oplus a^+_{i,j-1} \otimes \delta_{\{a^+_{i,j-1} > \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}\}}. \tag{14}$$

Let C be a $\mathbb{K}$-monotone matrix such that $A \leq_{\mathbb{K}} C$. The inequality $A^+ \leq_{\mathbb{K}} C$ is proved by induction on the column number. Since $a^+_{\cdot,1} = a_{\cdot,1}$, we obviously have $a^+_{\cdot,1} \leq_K c_{\cdot,1}$. Now, assume that for some $j \geq 2$,

$$\begin{pmatrix} a^+_{\cdot,1} & \cdots & a^+_{\cdot,j-1} \end{pmatrix} \leq_{\mathbb{K}} \begin{pmatrix} c_{\cdot,1} & \cdots & c_{\cdot,j-1} \end{pmatrix}.$$

The jth column of C satisfies

$$i = 1, \ldots, \eta : \qquad \mathbb{k}_{i,\cdot} \otimes a_{\cdot,j} \leq \mathbb{k}_{i,\cdot} \otimes c_{\cdot,j} \quad \text{and} \quad \mathbb{k}_{i,\cdot} \otimes c_{\cdot,j-1} \leq \mathbb{k}_{i,\cdot} \otimes c_{\cdot,j}.$$

This is equivalent to

$$i = 1, \ldots, \eta \qquad \mathbb{k}_{i,\cdot} \otimes c_{\cdot,j} \geq \mathbb{k}_{i,\cdot} \otimes a_{\cdot,j} \oplus \mathbb{k}_{i,\cdot} \otimes c_{\cdot,j-1}.$$

Since $a^+_{\cdot,j-1} \leq_{\mathbb{K}} c_{\cdot,j-1}$ by the induction assumption, we have

$$i = 1, \ldots, \eta \qquad \mathbb{k}_{i,\cdot} \otimes c_{\cdot,j} \geq \mathbb{k}_{i,\cdot} \otimes a_{\cdot,j} \oplus \mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j-1}.$$

But, we show now that the right hand side member of the last inequality is $\mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j}$. Thus, the induction will be complete.

Let us show that, for any $i, j \in S$, $\mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j} = \mathbb{k}_{i,\cdot} \otimes a_{\cdot,j} \oplus \mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j-1}$.
The proof is by induction on the row number. From the definition of A^+, we have $a^+_{\eta,j} = a_{\eta,j} \oplus a^+_{\eta,j-1}$, so that the result is true for $i = \eta$. Suppose that $\mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j} = \mathbb{k}_{i+1,\cdot} \otimes a_{\cdot,j} \oplus \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j-1}$ for some $i < \eta$. Noticing that

$$\mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j} \geq \mathbb{k}_{i,\cdot} \otimes a_{\cdot,j} \oplus \mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j-1},$$

we just have to justify that $\mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j} \leq \mathbb{k}_{i,\cdot} \otimes a_{\cdot,j} \oplus \mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j-1}$. Let us develop the following computation

$$\begin{array}{rcl}
\mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j} & = & a^+_{i,j} \oplus \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j} \text{ (by definition of } \mathbb{k}_{i,\cdot}) \\
& = & \big(a_{i,j} \otimes \delta_{\{a_{i,j} > \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}\}} \oplus a^+_{i,j-1} \otimes \delta_{\{a^+_{i,j-1} > \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}\}}\big) \oplus \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j} \\
& & \text{(from Definition (14) of } a^+_{i,j}) \\
& = & \big(a_{i,j} \otimes \delta_{\{a_{i,j} > \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}\}} \oplus \mathbb{k}_{i+1,\cdot} \otimes a_{\cdot,j}\big) \\
& & \oplus \big(a^+_{i,j-1} \otimes \delta_{\{a^+_{i,j-1} > \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}\}} \oplus \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j-1}\big) \\
& & \text{(by assumption on } \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}).
\end{array}$$

We get from $\delta_{\{\cdot\}} \leq \mathbb{1}$, $\mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j} \leq \mathbb{k}_{i,\cdot} \otimes a_{\cdot,j} \oplus \mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j-1}$. This last inequality ends the proof. ∎

Example 4.2
To illustrate the previous results, we consider an $(\oplus, \otimes)$-linear system with state index set $S = \{1, 2, 3, 4, 5\}$, where the dynamics is governed by the matrix

$$A = \left(\begin{array}{cc|ccc}
2 & 4 & \mathbb{1} & 3 & \mathbb{0} \\
-10 & 15 & -8 & \mathbb{0} & 20 \\
\hline
\mathbb{0} & \mathbb{0} & -1 & -9 & 1 \\
\mathbb{1} & 4 & \mathbb{0} & 7 & 2 \\
-7 & 4 & 2 & -10 & 8
\end{array}\right) \tag{15}$$

The monotone upper bound A^+ on A is obtained following the lines of the proof of Theorem 4.1

$$A^+ = \left(\begin{array}{cc|ccc}
2 & \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\
-10 & 15 & 15 & 15 & 20 \\
\hline
\mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\
\mathbb{1} & \mathbb{0} & \mathbb{0} & 7 & \mathbb{0} \\
-7 & 4 & 4 & 4 & 8
\end{array}\right) \tag{16}$$

4.2 Lower bound

The result for the monotone lower bound is based on the following lemma. Its proof follows that of Lemma 4.1 and is left to the reader.

Lemma 4.2 *Let a, b, c be three elements of $\mathbb{R}_{\max}$. Let us consider the system of inequalities $\mathcal{L}\big(a, b, c\big)$ defined by*

$$\begin{cases} y \oplus c \leq a \oplus b \\ c \leq b. \end{cases} \tag{17}$$

Then the solution set of $\mathcal{L}\big(a, b, c\big)$ is $\big[\mathbb{0}, y^+(a, b, c)\big]$ where $y^+(a, b, c) = a \oplus b$.

Theorem 4.3 (optimal $\mathbb{K}$-monotone lower bound) *Let A be an element of $\mathbb{R}_{\max}^{\eta\times\eta}$. Then there exists a matrix $A^- \in \mathbb{R}_{\max}^{\eta\times\eta}$ such that*

$$\begin{array}{ll} (a) & A^- \leq_{\mathbb{K}} A \\ (b) & A^- \textit{ is } \mathbb{K}\textit{-monotone} \\ (c) & \textit{for any monotone } C \textit{ such that } C \leq_{\mathbb{K}} A, \textit{ we have } C \leq_{\mathbb{K}} A^-. \end{array} \tag{18}$$

Proof. System (18) may be rewritten as (see Theorem 3.2 for (b))

$$\begin{array}{ll} (a) & \text{for } j = 1, \ldots, \eta, \quad \mathbb{K} \otimes a^-_{\cdot,j} \leq \mathbb{K} \otimes a_{\cdot,j} \\ (b) & \text{for } j = 1, \ldots, \eta - 1 \quad \mathbb{K} \otimes a^-_{\cdot,j} \leq \mathbb{K} \otimes a^-_{\cdot,j+1} \\ (c) & \text{for any monotone } C \text{ verifying } C \leq_{\mathbb{K}} A, \text{ we have } \mathbb{K} \otimes c_{\cdot,j} \leq \mathbb{K} \otimes a^-_{\cdot,j}, \\ & j = 1, \ldots, \eta. \end{array}$$

Once again, the construction of matrix A^- is by induction on the column number $j \in S$, starting with $a^-_{\cdot,\eta} = a_{\cdot,\eta}$.
For every column j, we have to solve the following constraints

$$i = 1, \ldots, \eta : \quad \mathbb{k}_{i,\cdot} \otimes a^-_{\cdot,j} \leq \mathbb{k}_{i,\cdot} \otimes a_{\cdot,j} \quad \text{and} \quad \mathbb{k}_{i,\cdot} \otimes a^-_{\cdot,j} \leq \mathbb{k}_{i,\cdot} \otimes a^-_{\cdot,j+1}.$$

If we assume that $\mathbb{k}_{i+1,\cdot} \otimes a^-_{\cdot,j}$, $a^-_{i,j+1}$ and $\mathbb{k}_{i+1,\cdot} \otimes a^-_{\cdot,j+1}$ are known, then we have to find a solution $a^-_{i,j}$ of

$$\mathcal{L}(a_{i,j}\,,\, \mathbb{k}_{i+1,\cdot} \otimes a_{\cdot,j}\,,\, \mathbb{k}_{i+1,\cdot} \otimes a^-_{\cdot,j}) \quad \text{and} \quad \mathcal{L}(a^-_{i,j+1}\,,\, \mathbb{k}_{i+1,\cdot} \otimes a^-_{\cdot,j+1}\,,\, \mathbb{k}_{i+1,\cdot} \otimes a^-_{\cdot,j}).$$

From Lemma 4.2, a maximal solution is given by

$$\begin{aligned} a^-_{i,j} &= \min\big(y^+(a_{i,j}\,,\, \mathbb{k}_{i+1,\cdot} \otimes a_{\cdot,j}\,,\, \mathbb{k}_{i+1,\cdot} \otimes a^-_{\cdot,j})\,,\, y^+(a^-_{i,j+1}\,,\, \mathbb{k}_{i+1,\cdot} \otimes a^-_{\cdot,j+1}\,,\, \mathbb{k}_{i+1,\cdot} \otimes a^-_{\cdot,j})\big) \\ &= \min\big(\mathbb{k}_{i,\cdot} \otimes a_{\cdot,j}\,,\, \mathbb{k}_{i,\cdot} \otimes a^-_{\cdot,j+1}\big). \end{aligned}$$

The optimality of the solution could be proved as for Theorem 4.1. ■

Without loss of generality, we can assume that A is a $\mathbb{K}$-monotone matrix till the end of the paper.

Example 4.4 (Example 4.2 continued)
Construction of a monotone lower bound for the matrix A is as in the proof of Theorem 4.3. This gives the following matrix A^{-}

$$A^{-} = \left(\begin{array}{cc|ccc} 2 & 2 & 2 & 7 & \mathbb{0} \\ \mathbb{1} & 2 & 2 & 7 & 20 \\ \hline \mathbb{1} & 2 & 2 & 7 & 1 \\ \mathbb{1} & 2 & 2 & 7 & 2 \\ -10 & -10 & -10 & -10 & 8 \end{array} \right) \tag{19}$$

5 Bounding the aggregated dynamics

Let us consider a lumping map ϕ from S into Σ, and V the corresponding lumping matrix (see Relation (4)). We can define a partition of S into N aggregates $\phi^{-1}(J) = [m_J, M_J]$ such that $\mathrm{cardinal}(\phi^{-1}(J)) = \eta_J$, $J \in \Sigma$. Additional notations are needed. For a matrix $X \in \mathbb{R}_{\max}^{\eta\times\eta}$, set $X^{I,J} = [x_{i,j}]_{i\in\phi^{-1}(I), j\in\phi^{-1}(J)}$ and $X^{\cdot,J} = [x_{i,j}]_{i\in S, j\in\phi^{-1}(J)}$. $x_{i,\cdot}^{I,J}$, $x_{\cdot,k}^{I,J}$, $x_{l,\cdot}^{\cdot,J}$, $x_{\cdot,j}^{\cdot,J}$, denote the ith row of matrix $X^{I,J}$, the kth column of matrix $X^{I,J}$, the lth row of matrix $X^{\cdot,J}$, the jth column of matrix $X^{\cdot,J}$ respectively. We recall that $x_{i,\cdot}^{I,J}$ and $x_{l,\cdot}^{\cdot,J}$ are considered as row-vectors. The scalar $x_{l,k}^{I,J}$ refers to the entry x_{m_I-1+l, m_J-1+k} of matrix $X = [x_{i,j}]_{i,j\in S}$.

The aim of this section is to find $\mathbb{K}$-bounds on the series $< y(n) >_{n=0}^{+\infty}$, which is defined by the following system

$$\begin{cases} x(n+1) & = A \otimes x(n) \\ y(n) & = V \otimes x(n). \end{cases}$$

where $x(n) \in \mathbb{R}_{\max}^{\eta\times 1}$, $y(n) \in \mathbb{R}_{\max}^{N\times 1}$ and $A \in \mathbb{R}_{\max}^{\eta\times\eta}$.

The series $< x(n) >_{n=0}^{+\infty}$ with given initial data $x(0)$, is said to be *lumpable* if the aggregated series $< y(n) >_{n=0}^{+\infty}$ satisfy the reduced equation

$$y(n+1) = \widehat{A} \otimes y(n) \tag{20}$$

for some $(N\times N)$-dimensional matrix $\widehat{A}$. In such a case, $< y(n) >_{n=0}^{+\infty}$ may be considered as an autonomous dynamics on $\mathbb{R}_{\max}^{N}$ governed by matrix $\widehat{A}$.

If there exist matrices L and U such that

$$L \leq_{\mathbb{K}} A \leq_{\mathbb{K}} U$$

and $l(0) \leq_{\mathbb{K}} x(0) \leq_{\mathbb{K}} u(0)$, then we have from Theorem 3.2

$$\forall n \geq 0, \quad l(n) \stackrel{\text{def}}{=} L^{\otimes n} \otimes l(0) \leq_{\mathbb{K}} x(n) \leq_{\mathbb{K}} u(n) \stackrel{\text{def}}{=} U^{\otimes n} \otimes u(0). \tag{21}$$

Additionally, assume that L, U are lumpable with corresponding matrices $\widehat{L}$ and $\widehat{U}$ respectively. The aggregated dynamics $< V \otimes l(n) >_{n=0}^{+\infty}$ and $< V \otimes u(n) >_{n=0}^{+\infty}$ are lower and upper $\mathbb{K}$-bounds for the aggregated series $< y(n) >_{n=0}^{+\infty}$. Indeed, since ϕ is non-decreasing, it follows from Inequalities (21) that

$$V \otimes l(n) \leq_{\mathbb{K}} y(n) \leq_{\mathbb{K}} V \otimes u(n).$$

Finally, the lumpability property will give that the aggregated dynamics $< V \otimes l(n) >_{n=0}^{+\infty}$ and $< V \otimes u(n) >_{n=0}^{+\infty}$ are governed by the matrices $\widehat{L}$ and $\widehat{U}$ respectively, i.e.

$$\forall n \geq 0, \quad V \otimes l(n) = V \otimes \widehat{L}^{\otimes n} \otimes l(0) \quad V \otimes u(n) = V \otimes \widehat{U}^{\otimes n} \otimes u(0).$$

In the following subsections, we focus on two conditions to identify a lumpable matrix. For each condition, we show that any $\mathbb{K}$-monotone matrix A may be bounded from above and from below by a lumpable matrix. Thus, we get $\mathbb{K}$-bounds on the aggregated dynamics $< y(n) >_{n=0}^{+\infty}$. Similar methods were used for Markov chains in [7].

5.1 Strongly lumpable matrix

Definition 5.1 *$A \in \mathbb{R}_{\max}^{\eta\times\eta}$ is said to be strongly lumpable by V, or simply V-lumpable [8], if there exists $\widehat{A} \in \mathbb{R}_{\max}^{N\times N}$ such that $V \otimes A = \widehat{A} \otimes V$. Equivalently, this means*

$$\forall I \in \Sigma, \forall J \in \Sigma, \forall j \in \phi^{-1}(J) \qquad \bigoplus_{i\in\phi^{-1}(I)} a_{i,j} = \widehat{a}_{I,J}.$$

The lumped matrix $\widehat{A}$ is then $V \otimes A \otimes V^{\top}$.

When the autonomous dynamics $< x(n) >_{n=0}^{+\infty}$ is governed by a strongly lumpable matrix A, the aggregated variables $y(n) = V \otimes x(n)$ satisfy the autonomous difference equation (20). Indeed, we have

$$\begin{aligned} y(n+1) &= V \otimes x(n+1) = V \otimes A \otimes x(n) \\ &= \widehat{A} \otimes V \otimes x(n) \text{ (using Definition 5.1)} \\ &= \widehat{A} \otimes y(n). \end{aligned}$$

Theorem 5.1 *There always exist V-lumpable matrices U and L such that*

$$L \leq_{\mathbb{K}} A \leq_{\mathbb{K}} U. \tag{22}$$

Proof. Since A is $\mathbb{K}$-monotone, Inequality (22) holds for the following matrices $L = [l_{i,j}]$ and $U = [u_{i,j}]$

$$\forall I \in \Sigma, \forall J \in \Sigma: \quad u_{i,j}^{I,J} = a_{i,\eta_J}^{I,J}, \quad l_{i,j}^{I,J} = a_{i,1}^{I,J} \quad i = 1, \ldots \eta_I, \quad j = 1, \ldots, \eta_J. \tag{23}$$

It is easily seen from their definition that L and U are V-lumpable. ■

Example 5.2 (Example 4.2 continued)
The lumping map is $\phi : S \to \Sigma = \{1, 2\}$ where $\phi(1) = \phi(2) = 1$ et $\phi(3) = \phi(4) = \phi(5) = 2$. The corresponding matrix V is

$$V = \begin{pmatrix} \mathbb{1} & \mathbb{1} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} & \mathbb{1} & \mathbb{1} \end{pmatrix}$$

U, L denote the strongly lumpable upper and lower bounds for A^+ and A^- respectively. The method of construction of these matrices is given in the previous proof.

$$U = \left(\begin{array}{cc|ccc} \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ 15 & 15 & 20 & 20 & 20 \\ \hline \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ 4 & 4 & 8 & 8 & 8 \end{array}\right) \quad L = \left(\begin{array}{cc|ccc} 2 & 2 & 2 & 2 & 2 \\ \mathbb{1} & \mathbb{1} & 2 & 2 & 2 \\ \hline \mathbb{1} & \mathbb{1} & 2 & 2 & 2 \\ \mathbb{1} & \mathbb{1} & 2 & 2 & 2 \\ -10 & -10 & -10 & -10 & -10 \end{array}\right). \tag{24}$$

The corresponding aggregated $(\oplus, \otimes)$-systems, are governed by the matrices

$$\widehat{U} = V \otimes U \otimes V^\top = \begin{pmatrix} 15 & 20 \\ 4 & 8 \end{pmatrix} \quad \text{and } \widehat{L} = V \otimes L \otimes V^\top = \begin{pmatrix} 2 & 2 \\ \mathbb{1} & 2 \end{pmatrix}$$

respectively.

5.2 Coherency

Let us consider the $(\eta \times N)$-dimensional matrix $C = \mathrm{diag}(c^J)$, where, for $J = 1, \ldots, N$, vector $c^J \in \mathbb{R}^{\eta_J \times 1}$ is a normalized positive vector in the following sense

$$j = 1, \ldots, \eta_J \quad c_j^J > \mathbb{0} \quad \text{and} \quad \mathbb{1}_{\eta_J}^\top \otimes c^J = \mathbb{1}.$$

In particular, we have $V \otimes C = I_N$ where $I_N \stackrel{\text{def}}{=} (\delta_{\{I=J\}})_{I,J=1,\ldots,N}$.

Definition 5.2 *A matrix $A \in \mathbb{R}_{\max}^{\eta\times\eta}$ is C-coherent [8] w.r.t. the lumping map ϕ if there exists a matrix $\widehat{A} \in \mathbb{R}_{\max}^{N\times N}$ such that $A \otimes C = C \otimes \widehat{A}$ or*

$$\forall I, J \in \Sigma, \quad A^{I,J} \otimes c^J = \widehat{a}_{I,J} \otimes c^I. \tag{25}$$

In this case, the matrix $\widehat{A}$ is $V \otimes A \otimes C$.

When the autonomous dynamics $< x(n) >_{n=0}^{+\infty}$ is governed by a C-coherent matrix A, we have for any $x(0) \in \mathrm{Im}C \stackrel{\text{def}}{=} \{C \otimes u \mid u \in \mathbb{R}_{\max}^{N\times 1}\}$

$$n \geq 1, \quad x(n) = A^{\otimes n} \otimes C \otimes u = C \otimes \widehat{A}^{\otimes n} \otimes u.$$

Hence, the dynamics of the original model may be derived from that of the aggregated system. It also follows that the aggregated dynamics $< y(n) \stackrel{\text{def}}{=} V \otimes x(n) >_{n=0}^{+\infty}$ is an autonomous dynamics

$$\begin{aligned} y(n+1) &= V \otimes C \otimes \widehat{A}^{\otimes(n+1)} \otimes u = \widehat{A}^{\otimes(n+1)} \otimes u \text{ (since } V \otimes C = I_N) \\ &= \widehat{A} \otimes V \otimes C \otimes \widehat{A}^{\otimes n} \otimes u \\ &= \widehat{A} \otimes y(n). \end{aligned}$$

Remark 5.1 *Considering a normalized vector c^J in matrix C of Definition 5.2 is not a major restriction. Indeed, C-coherency may be defined from any set of positive vectors $c^J (J = 1, \ldots, N)$, i.e. $c^J \in \mathbb{R}^{\eta_J \times 1}$. Thus, we choose matrix C such that $V \otimes C = I_N$ for writing convenience.*

Example 5.3 (Example 5.2 continued)
We consider the matrix $C = diag(c^1, c^2)$ where $c^1 = \mathbb{1}_2{}^\top$ and $c^2 = \mathbb{1}_3{}^\top$. The following matrix W^+ denotes one of the upper C-coherent bounds on A^+

$$W^+ = \left(\begin{array}{cc|ccc} 15 & 2 & \mathbb{0} & 20 & \mathbb{0} \\ -10 & 15 & 15 & 15 & 20 \\ \hline \mathbb{0} & 4 & \mathbb{0} & \mathbb{0} & 8 \\ \mathbb{1} & 4 & \mathbb{0} & 7 & 8 \\ -7 & 4 & 4 & 4 & 8 \end{array}\right). \tag{26}$$

The dynamics of the aggregated $(\oplus, \otimes)$-system obtained from matrix W^+ is governed by the matrix

$$\widehat{W^+} = V \otimes W^+ \otimes C = \begin{pmatrix} 15 & 20 \\ 4 & 8 \end{pmatrix}$$

Note that, even if $\widehat{W^+} = \widehat{U}$ (see Example 5.2), W^+ is not strongly lumpable.

We will show that there is a counterpart to Theorem 5.1 in the context of coherency. We need the next lemma, which follows from [1, p 112].

Lemma 5.1 *Let* $a = (a_1, \ldots, a_n)^\top \in \mathbb{R}^{n\times 1}$ *and* $d \in \mathbb{R}_{\max}$ *be fixed. Then,*

$$\big(x \in \mathbb{R}_{\max}^{n\times 1} \quad \textit{and} \quad x^\top \otimes a \le d\big) \Longleftrightarrow x \le (d - a_i)_{i=1,\ldots,n}^\top \tag{27}$$

and we always have

$$(d - a_i)_{i=1,\ldots,n}^\top \otimes a = d. \tag{28}$$

Moreover, for any $b, d \in \mathbb{R}_{\max}$ *and* $c \in \mathbb{R}$

$$(b - c) \oplus (d - c) = (b \oplus d) - c. \tag{29}$$

The next theorem states that, for any monotone matrix A, there always exists a C-coherent upper bound. We emphasize that an explicit C-coherent upper bound will be given in the proof (see Formula (36)).

Theorem 5.4 *For any* $\mathbb{K}$*-monotone matrix* A*, there always exists a* C*-coherent matrix* U *such that*

$$A \le_{\mathbb{K}} U. \tag{30}$$

The corresponding aggregated matrix $\widehat{U} = [\widehat{u}_{I,J}] \in \mathbb{R}_{\max}^{N\times N}$ *has entries that are a solution of system*

$$\begin{array}{l} I, J = 1, \ldots, N : \\ \big(\mathbb{K}_{\eta_I} \otimes A^{I,J} \oplus \mathbb{1}_{\eta_I} \otimes [\oplus_{K=I+1}^N \mathbb{1}_{\eta_K}^\top \otimes A^{K,J}]\big) \otimes c^J \le \\ \widehat{u}_{I,J} \otimes \mathbb{K}_{\eta_I} \otimes c^I \oplus (\oplus_{K=I+1}^N \widehat{u}_{K,J}) \otimes \mathbb{1}_{\eta_I} \textit{ (component-wise)}. \end{array} \tag{31}$$

Proof.

Firstly, assume that there exists a C-coherent matrix U such that Inequality (30) holds. Let $\widehat{U}$ be the matrix associated with the C-coherent matrix U (see (25)). It is easily seen that $A \le_{\mathbb{K}} U$ iff

$$\forall J \in \{1, \ldots, N\} \qquad \mathbb{K}_\eta \otimes A^{\cdot,J} \le \mathbb{K}_\eta \otimes U^{\cdot,J}. \tag{32}$$

$\otimes$-right-multiplying this last inequality by the normalized vector c^J and using Relation (25), we obtain that the entries of $\widehat{U}$ satisfy System (31).

Secondly, let us show that System (31) has always a solution. This system may be rewritten as, for any $I, J \in \{1, \ldots, N\}$

$$\forall i \in \{1, \ldots \eta_I\}$$

$$\big(\mathbb{k}_{i,\cdot}^{I,I} \otimes A^{I,J} \oplus \bigoplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^\top \otimes A^{K,J}\big) \otimes c^J \le \widehat{u}_{I,J} \otimes \mathbb{k}_{i,\cdot}^{I,I} \otimes c^I \oplus [\bigoplus_{K=I+1}^{N} \widehat{u}_{K,J}]. \tag{33}$$

Now, let us fix $J \in \{1,\dots,N\}$. For $I = N$, we can set

$$\widehat{u}_{N,J} = \bigoplus_{i=1}^{\eta_N} \left(\mathbb{k}_{i,\cdot}^{N,N} \otimes A^{N,J} \otimes c^J - \mathbb{k}_{i,\cdot}^{N,N} \otimes c^N\right) \tag{34}$$

Assume that we have obtained $\widehat{u}_{K,J}$ for $K = I+1,\dots,N$ $(I < N)$. $\widehat{u}_{I,J}$ will be a solution of System (33) if $\widehat{u}_{I,J}$ satisfies the following system

$$\forall i \in \{1,\dots,\eta_I\} \quad \widehat{u}_{I,J} \geq \left(\mathbb{k}_{i,\cdot}^{I,I} \otimes A^{I,J} \oplus \bigoplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J}\right) \otimes c^J - \mathbb{k}_{i,\cdot}^{I,I} \otimes c^I.$$

Note that the right hand side member in the above inequalities is well defined, since $\mathbb{k}_{i,\cdot}^{I,I} \otimes c^I > \mathbb{0}$ ($c^I > \mathbb{0}$). Finally, we just have to set

$$\widehat{u}_{I,J} = \bigoplus_{i=1}^{\eta_I} \left((\mathbb{k}_{i,\cdot}^{I,I} \otimes A^{I,J} \oplus \bigoplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J}) \otimes c^J - \mathbb{k}_{i,\cdot}^{I,I} \otimes c^I\right). \tag{35}$$

Finally, let us give a C-coherent matrix U satisfying (30) from the matrix $\widehat{U}$ previously defined. Fix $J \in \Sigma$. For every $I = 1,\dots,N$, set

$$u_{i,\cdot}^{I,J} = \left(\widehat{u}_{I,J} \otimes c_i^I - c_j^J\right)_{j=1,\dots,\eta_J}^{\top} \quad i = 1,\dots,\eta_I. \tag{36}$$

Let us check that U is a C-coherent matrix. We have to prove Relation (25), i.e.

$$\forall I, J \in \Sigma,\ \forall i \in \{1,\dots,\eta_I\} \qquad u_{i,\cdot}^{I,J} \otimes c^J = \widehat{u}_{I,J} \otimes c_i^I.$$

This is clear from Definition (36) of vector $u_{i,\cdot}^{I,J}$ and from Relation (28) (with $d = \widehat{u}_{I,J} \otimes c_i^I$ and $a = c^J$).
It remains to show that $A^{\cdot,J} \leq_{\mathbb{K}} U^{\cdot,J}$, i.e.

$$\forall i \in \{1,\dots,\eta\} \qquad m(i) \overset{\text{def}}{=} \bigoplus_{k=i}^{\eta} a_{k,\cdot}^{\cdot,J} \leq \bigoplus_{k=i}^{\eta} u_{k,\cdot}^{\cdot,J}. \tag{37}$$

Let us define scalar r_i as follows

$$r_i = \widehat{u}_{\phi(i),J} \otimes \left[e_{\eta_{\phi(i)}}^{\top}(i - a_{\phi(i)} + 1) \otimes \mathbb{K}_{\eta_{\phi(i)}} \otimes c^{\phi(i)}\right] \oplus \bigoplus_{K=\phi(i)+1}^{N} \widehat{u}_{K,J} \tag{38}$$

where $e_{\eta_{\phi(i)}}(j)$ is the vector $(\delta_{\{k=j\}})_{k=1,\dots,\eta_{\phi(i)}}$.
It is easily checked that System (31) for fixed J, is

$$i = 1,\dots,\eta \qquad m(i) \otimes c^J \leq r_i. \tag{39}$$

Moreover, we have from Definition (36) of $U^{I,J}$ and Equality (29)

$$i = 1, \dots, \eta \qquad \bigoplus_{k=i}^{\eta} u_{k,\cdot}^{\cdot,J} = \left(r_i - c_j^J\right)_{j=1,\dots,\eta_J}^{\top}. \tag{40}$$

Applying Relation (27) to solve Inequality (39) with $a = c^J$, $x^\top = m(i)$ and $d = r_i$ for each $i = 1, \dots, \eta$, we get

$$j = 1, \dots, \eta_J \qquad m_j(i) \leq r_i - c_j^J \underset{(40)}{=} \bigoplus_{k=i}^{\eta} u_{k,j}^{\cdot,J}.$$

The proof is complete. ■

Remark 5.2 *We can derive another solution $\widehat{u}_{I,J}$ of system (33). Indeed, Formula (35) $(1 \leq I < N)$ can be replaced by*

$$\widehat{u}_{I,J} = \begin{cases} \bigoplus_{i \in G_{I,J}} \left(\left(\mathbb{k}_{i,\cdot}^{I,I} \otimes A^{I,J} \oplus \bigoplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J} \right) \otimes c^J - \mathbb{k}_{i,\cdot}^{I,I} \otimes c^I \right) & \textit{if } G_{I,J} \neq \emptyset \\ \mathbb{0} & \textit{if } G_{I,J} = \emptyset \end{cases} \tag{41}$$

where $G_{I,J} = \left\{ i \in \{1, \dots, \eta_I\} \mid \oplus_{K=I+1}^{N} \widehat{u}_{K,J} < \left(\mathbb{k}_{i,\cdot}^{I,I} \otimes A^{I,J} \oplus \oplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J} \right) \otimes c^J \right\}$.

Remark 5.3 *We emphasize that we get a C-coherent upper bound, whatever the choice of matrix C. Thus, the problem of the selection of an appropriate matrix C for having such a C-coherent bound does not arise. The same remark holds for the lower bounds.*

Example 5.5 (Example 5.3 continued)
Consider the matrix $C = \operatorname{diag}(c^1, c^2)$, where $c^1 = (\mathbb{1}\,,\, -3)^\top$, $c^2 = (-12\,,\, \mathbb{1}\,,\, -4)^\top$, and

$$\mathbb{K}_2 \otimes c^1 = (\mathbb{1}\,,\, -3)^\top, \qquad \mathbb{K}_3 \otimes c^2 = (\mathbb{1}\,,\, \mathbb{1}\,,\, -4)^\top.$$

Using Formulae (34), (35), we obtain as matrix $\widehat{U}$

$$\widehat{U} = \begin{pmatrix} 15 & 19 \\ 5 & 8 \end{pmatrix}. \tag{42}$$

We get from (36) the following C-coherent matrix U such that $A \leq_{\mathbb{K}} A^+ \leq_{\mathbb{K}} U$

$$U = \left(\begin{array}{cc|ccc} 15 & 18 & 31 & 19 & 23 \\ 12 & 15 & 28 & 16 & 20 \\ \hline -7 & -4 & 8 & -4 & \mathbb{1} \\ 5 & 8 & 20 & 8 & 12 \\ 1 & 4 & 16 & 4 & 8 \end{array}\right).$$

Let us choose $x(0) = (2\,,\,-3\,;\,2\,,\,4\,,\,-15)^\top$ for the $(\oplus,\otimes)$-linear system governed by the matrix A in Example 4.2. In Table 1, we report the dynamics of $< x(n) >_{n=0}^{3}$ and that of the corresponding reduced series $< y(n) >_{n=0}^{3}$.

step	$y(n) = V \otimes x(n)$	$x(n) = A^{\otimes\, n} \otimes x(0)$
0	$(2\,;\,4)^\top$	$(2\,,\,-3\,;\,2\,,\,4\,,\,-15)^\top$
1	$(12\,;\,11)^\top$	$(7\,,\,12\,;\,1\,,\,11\,,\,4)^\top$
2	$(27\,;\,18)^\top$	$(16\,,\,27\,;\,5\,,\,18\,,\,16)^\top$
3	$(42\,;\,31)^\top$	$(31\,,\,42\,;\,17\,,\,31\,,\,31)^\top$

Table 1: The dynamics of the system $(x(0), A)$.

If $\widehat{u}(0) = (\mathbb{1}, 4)$, then we have

$$x(0) \leq_{\mathbb{K}} u(0) = C \otimes \widehat{u}(0) = \big((c^1)^\top\,;\,4 \otimes (c^2)^\top\big)^\top.$$

From $\widehat{U}$ defined by Equation (42), we deduce the dynamics of the aggregated and original systems associated with the upper bound U of A. This gives upper $\mathbb{K}$-bounds on series $< y(n) >_{n=0}^{+\infty}$ and $< x(n) >_{n=0}^{+\infty}$ respectively.

step	$\widehat{u}(n) = \widehat{U}^{\otimes\, n} \otimes \widehat{u}(0)$	$u(n) = U^{\otimes\, n} \otimes u(0) = C \otimes \widehat{u}(n)$
0	$(\mathbb{1}\,;\,4)^\top$	$(\mathbb{1}\,,\,-3\,;\,-8\,,\,4\,,\,\mathbb{1})^\top$
1	$(23\,;\,12)^\top$	$(23\,,\,20\,;\,\mathbb{1}\,,\,12\,,\,8)^\top$
2	$(38\,;\,28)^\top$	$(38\,,\,35\,;\,16\,,\,28\,,\,24)^\top$
3	$(53\,;\,43)^\top$	$(53\,,\,50\,;\,31\,,\,43\,,\,39)^\top$

Table 2: The dynamics of systems $(\widehat{u}(0), \widehat{U})$ and $(C \otimes \widehat{u}(0), U)$.

It is easily checked that there always exists a C-coherent lower bound L for a $\mathbb{K}$-monotone matrix A. Indeed, set $L = (\mathbb{0})$. However, we can obtain another (non-trivial) lower bound. We need the properties reported in the next lemma. Its proof is similar to that of [1, Th 3.21]. For any $a, b \in \mathbb{R}_{\max}$, $a \wedge b$ stands for $\min(a, b)$. The operator $\wedge$ is assumed to have the same priority than $\oplus$ w.r.t. $\otimes$. If $A \in \mathbb{R}_{\max}^{n\times p}$ and $B \in \mathbb{R}_{\max}^{p\times q}$, the product $A \wedge B$ is defined by

$$A \wedge B \stackrel{\text{def}}{=} \left[\bigwedge_{k=1}^{p} a_{i,k} \otimes b_{k,j} = \min_{k=1,\ldots,p}(a_{i,k} + b_{k,j})\right]_{i=1,\ldots,n;j=1,\ldots,q}$$

Lemma 5.2 *Let $a = (a_1, \ldots, a_n)^\top \in \mathbb{R}^{n\times 1}$ and $d \in \mathbb{R}_{\max}$ be fixed. Then,*

$$\big(x \in \mathbb{R}_{\max}^{n\times 1} \text{ and } x^\top \wedge a \geq d\big) \Longleftrightarrow x \geq (d - a_i)_{i=1,\ldots,n}^\top. \tag{43}$$

We also have

$$(d - a_i)_{i=1,\ldots,n}^\top \wedge a = d. \tag{44}$$

Now, we present our result for the lower bound.

Theorem 5.6 *For any $\mathbb{K}$-monotone matrix A, there always exists a C-coherent matrix L such that*

$$L \leq_{\mathbb{K}} A. \tag{45}$$

The corresponding aggregated matrix $\widehat{L} = [\widehat{l}_{I,J}] \in \mathbb{R}_{\max}^{N\times N}$ has entries that are a solution of system

$$\begin{array}{l} I, J = 1, \ldots, N: \\ \widehat{l}_{I,J} \otimes \mathbb{K}_{\eta_I} \otimes c^I \oplus [\oplus_{K=I+1}^{N} \widehat{l}_{K,J}] \otimes \mathbb{1}_{\eta_I} \leq \\ \big(\mathbb{K}_{\eta_I} \otimes A^{I,J} \oplus \mathbb{1}_{\eta_I} \otimes [\oplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J}]\big) \wedge c^J \textit{ (component-wise)}. \end{array} \tag{46}$$

Proof.
Firstly, let us show that System (46) has always a solution. For $I = N$, we have to solve

$$i = 1, \ldots, \eta_N, \qquad \widehat{l}_{N,J} \otimes \mathbb{k}_{i,\cdot}^{N,N} \otimes c^N \leq (\mathbb{k}_{i,\cdot}^{N,N} \otimes A^{N,J}) \wedge c^J.$$

So, we can set

$$\widehat{l}_{N,J} = \bigwedge_{i=1}^{\eta_N} \big((\mathbb{k}_{i,\cdot}^{N,N} \otimes A^{N,J}) \wedge c^J - \mathbb{k}_{i,\cdot}^{N,N} \otimes c^N\big). \tag{47}$$

Note that we have (with $i = 1$), $\widehat{l}_{N,J} \leq (\mathbb{1}_{\eta_N}^{\top} \otimes A^{N,J}) \wedge c^J$ since $\mathbb{1}_{\eta_N}^{\top} \otimes c^N = \mathbb{1}$. Suppose now that we have obtained $\widehat{l}_{K,J}$ for $K = I+1, \ldots, N$ $(I < N)$ and

$$\bigoplus_{K=I+1}^{N} \widehat{l}_{K,J} \leq \big(\bigoplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J}\big) \wedge c^J \tag{48}$$

We must derive $\widehat{l}_{I,J}$ from (46), i.e.

$$\begin{gathered} i = 1, \ldots, \eta_I, \\ \widehat{l}_{I,J} \otimes \mathbb{k}_{i,\cdot}^{I,I} \otimes c^I \oplus \bigoplus_{K=I+1}^{N} \widehat{l}_{K,J} \leq \big(\mathbb{k}_{i,\cdot}^{I,I} \otimes A^{I,J} \oplus \bigoplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J}\big) \wedge c^J \end{gathered} \tag{49}$$

It follows from $\oplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J} \leq \mathbb{k}_{i,\cdot}^{I,I} \otimes A^{I,J} \oplus \oplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J}$, that

$$\begin{aligned} \big(\mathbb{k}_{i,\cdot}^{I,I} \otimes A^{I,J} \oplus \oplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J}\big) \wedge c^J &\geq \big(\bigoplus_{K=I+1}^{N} \mathbb{1}_{\eta_K}^{\top} \otimes A^{K,J}\big) \wedge c^J \\ &\geq \bigoplus_{K=I+1}^{N} \widehat{l}_{K,J} \quad \text{from (48).} \end{aligned}$$

Hence, solving system (49) is equivalent to solve

$$i = 1, \ldots, \eta_I,$$
$$\widehat{l}_{I,J} \otimes \mathbb{k}^{I,I}_{i,\cdot} \otimes c^I \leq \big(\mathbb{k}^{I,I}_{i,\cdot} \otimes A^{I,J} \oplus \bigoplus_{K=I+1}^{N} \mathbb{1}^{\top}_{\eta_K} \otimes A^{K,J}\big) \wedge c^J.$$

We set

$$\widehat{l}_{I,J} = \bigwedge_{i=1}^{\eta_N} \Bigg(\big(\mathbb{k}^{I,I}_{i,\cdot} \otimes A^{I,J} \oplus \bigoplus_{K=I+1}^{N} \mathbb{1}^{\top}_{\eta_K} \otimes A^{K,J}\big) \wedge c^J - \mathbb{k}^{I,I}_{i,\cdot} \otimes c^I \Bigg). \tag{50}$$

In particular, we have $\widehat{l}_{I,J} \leq \big(\bigoplus_{K=I}^{N} \mathbb{1}^{\top}_{\eta_K} \otimes A^{K,J}\big) \wedge c^J$ since $\mathbb{1}^{\top}_{\eta_I} \otimes c^I = \mathbb{1}$. We deduce from (48) that

$$\bigoplus_{K=I+1}^{N} \widehat{l}_{K,J} \leq \big(\bigoplus_{K=I+1}^{N} \mathbb{1}^{\top}_{\eta_K} \otimes A^{K,J}\big) \wedge c^J \leq \big(\bigoplus_{K=I}^{N} \mathbb{1}^{\top}_{\eta_K} \otimes A^{K,J}\big) \wedge c^J$$

Therefore, we obtain that

$$\bigoplus_{K=I}^{N} \widehat{l}_{K,J} \leq \big(\bigoplus_{K=I}^{N} \mathbb{1}^{\top}_{\eta_K} \otimes A^{K,J}\big) \wedge c^J.$$

Secondly, from $\widehat{l}_{I,J}$ $I, J = 1, \ldots, N$ satisfying System (46), we define a C-coherent matrix L such that Inequality (45) holds as follows. Fix $J \in \Sigma$. For every $I = 1, \ldots, N$, set

$$l^{I,J}_{i,\cdot} = \Big(\widehat{l}_{I,J} \otimes c^I_i - c^J_j \Big)^{\top}_{j=1,\ldots,\eta_J} \qquad i = 1, \ldots, \eta_I. \tag{51}$$

First, it is easily seen from the definition (51) of matrix L and Equation (28) that, for each $I \in \Sigma$ and for all $i \in \{1, \ldots, \eta_I\}$

$$l^{I,J}_{i,\cdot} \otimes c^J = \widehat{l}_{I,J} \otimes c^I_i.$$

Thus, matrix L is C-coherent. Note that we also have from Equality (44)

$$l^{I,J}_{i,\cdot} \wedge c^J = \widehat{l}_{I,J} \otimes c^I_i = l^{I,J}_{i,\cdot} \otimes c^J$$

Second, we must show that $L^{\cdot,J} \leq_{\mathbb{K}} A^{\cdot,J}$, i.e.

$$i = 1, \ldots, \eta \qquad \bigoplus_{k=i}^{\eta} l^{\cdot,J}_{k,\cdot} \leq m(i) \stackrel{\text{def}}{=} \bigoplus_{k=i}^{\eta} a^{\cdot,J}_{k,\cdot}. \tag{52}$$

We make the following remarks. System (46) could also be written

$$i = 1, \dots, \eta \qquad s_i \le m(i) \wedge c^J, \tag{53}$$

where $s_i \in \mathbb{R}_{\max}$ is defined by ($i = 1, \dots, \eta$)

$$s_i = \left[e^\top_{\eta_{\phi(i)}}(i - a_{\phi(i)} + 1) \otimes \mathbb{K}_{\eta_{\phi(i)}} \otimes c^{\phi(i)} \right] \otimes \widehat{l}_{\phi(i),J} \oplus \bigoplus_{K=\phi(i)+1}^{N} \widehat{l}_{K,J}, \tag{54}$$

and $e_{\eta_{\phi(i)}}(j)$ is the vector $(\delta_{\{k=j\}})_{k=1,\dots,\eta_{\phi(i)}}$.
From the definition of L and Equality (29), we have

$$i = 1, \dots, \eta \qquad \bigoplus_{k=i}^{\eta} l^{\cdot,J}_{k,\cdot} = \left(s_i - c^J_j \right)^\top_{j=1,\dots,\eta_J}. \tag{55}$$

From these results, we just have to apply Formula (43) with $a = c^J$, $x^\top = m(i)$ and $d = s_i$ for each $i = 1, \dots, \eta$, to Inequality (53). Thus, we get

$$j = 1, \dots, \eta_J \qquad m_j(i) \ge s_i - c^J_j \underset{(55)}{=} \bigoplus_{k=i}^{\eta} l^{\cdot,J}_{k,j},$$

and the proof is complete. ∎

Remark 5.4 *We emphasize that an explicit C-coherent lower bound L is given by Formula (51)). Note that this definition provides a C-lumpable matrix in the max-plus algebra, which is also C-lumpable in the min-plus algebra.*

Example 5.7 (Example 5.5 continued)
Matrix C and vector $x(0)$ are as in Example 5.5. We get from Formulae (47) and (50)

$$\widehat{L} = \begin{pmatrix} -1 & -10 \\ -9 & -18 \end{pmatrix}.$$

The expanded matrix L of $\widehat{L}$ such that $L \le_{\mathbb{K}} A^- \le_{\mathbb{K}} A$, is from Formula (51)

$$L = \left(\begin{array}{cc|ccc} -1 & 2 & 2 & -10 & -6 \\ -4 & -1 & -1 & -13 & -9 \\ \hline -21 & -18 & -18 & -30 & -26 \\ -9 & -6 & -6 & -18 & -14 \\ -13 & -10 & -10 & -22 & -18 \end{array} \right).$$

If $\widehat{l}(0) = (\mathbb{0}, -11)^\top$, then

$$l(0) = C \otimes \widehat{l}(0) = \left((c^1)^\top \otimes \mathbb{0}\,;\,(c^2)^\top \otimes -11\right)^\top \leq_{\mathbb{K}} x(0).$$

In Table 3, we report the dynamics of the aggregated and original systems associated with the lower bound L of A. The dynamics of the original system is computed from $\widehat{L}$. This gives lower $\mathbb{K}$-bounds on $< y(n) >_{n=0}^{3}$ and $< x(n) >_{n=0}^{3}$.

step	$\widehat{l}(n) = \widehat{L}^{\otimes n} \otimes \widehat{l}(0)$	$l(n) = L^{\otimes n} \otimes l(0) = C \otimes \widehat{l}(n)$
0	$(\mathbb{0}\,;\,-11)^\top$	$(\mathbb{0}\,,\,\mathbb{0}\,;\,-23\,;\,-11\,,\,-15)^\top$
1	$(-21\,;\,-29)^\top$	$(-21\,,\,-24\,;\,-41\,,\,-29\,,\,-33)^\top$
2	$(-22\,;\,-30)^\top$	$(-22\,,\,-33\,;\,-42\,,\,-30\,,\,-34)^\top$
3	$(-23\,;\,-31)^\top$	$(-23\,,\,-26\,;\,-43\,,\,-31\,,\,-35)^\top$

Table 3: The dynamics of systems $(\widehat{l}(0), \widehat{L})$ and $(C \otimes \widehat{l}(0), L)$.

6 Algorithms

In this section, we report the algorithms associated with the bounds provided by Theorems 4.1, 5.1, 5.4. We only deal with the case of upper bounds. Lower bounds are obtained in a similar way. Let us consider an autonomous dynamics governed by matrix A. Algorithm **UpOpt** allows one to get a $\mathbb{K}$-monotone upper bound A^+ on A. Next, we consider the aggregated dynamics w.r.t. some lumping map. Two algorithms that compute bounds on this aggregated dynamics are presented. The first algorithm uses the construction of a strongly lumpable bound on A. The second algorithm provides a bound that is derived from a C-lumpable upper bound on A. Finally, we address their complexity.

Let us recall that $\phi^{-1}(I) = [m_I, M_I]$, for $I = 1, \ldots, N$ and $A = [a_{i,j}]_{i,j \in S}$. **UpOpt**($a_{\cdot,j}$) is the function that returns the optimal (in the sense defined in Theorem 4.1) column $a^+_{\cdot,j}$ from a column $a_{\cdot,j}$ of A such that: (a) $a^+_{\cdot,j-1} \leq_{\mathbb{K}} a^+_{\cdot,j}$ and (b) $a_{\cdot,j} \leq_{\mathbb{K}} a^+_{\cdot,j}$ (with the convention that property (a) holds when $j = 1$). From Formula (14), and using the relation

$$\mathbb{k}_{i,\cdot} \otimes a^+_{\cdot,j} = a^+_{i,j} \oplus \mathbb{k}_{i+1,\cdot} \otimes a^+_{\cdot,j}$$

we get

UpOpt($a_{\cdot,j}$)
$\alpha := \mathbb{0}$
For $i = \eta$ to 1
Begin

$a^+_{i,j} := a_{i,j} \otimes \delta_{\{a_{i,j}>\alpha\}} \oplus a^+_{i,j-1} \otimes \delta_{\{a^+_{i,j-1}>\alpha\}}$
$\alpha := \alpha \oplus a^+_{i,j}$
End
Return $a^+_{\cdot,j}$

Construction of a $\mathbb{K}$-monotone upper bound on A

Let V be the matrix associated with the considered lumping map from S into Σ (see (4)). Using Formula (23), we derive now an upper bound $\widehat{U}$ on the aggregated dynamics specified by V.

Strong(A, V)
For $J = 1$ to N
Begin
For $j = m_J$ to M_J (* loop UP *)
Begin
Generate($a_{\cdot,j}$)
If $j = 1$ then $a^+_{\cdot,j} := a_{\cdot,j}$
else $a^+_{\cdot,j}$:= **UpOpt**($a_{\cdot,j}$)
Free($a^+_{\cdot,j-1}$, $a_{\cdot,j}$)
End
$\widehat{u}_{\cdot,J} := V \otimes a^+_{\cdot,M_J}$
End
Return $\widehat{U}$

Construction of an upper bound $\widehat{U}$ based on the strong lumpability

Under this form, **UpOpt** has a time complexity in $O(\eta)$. Using the particular structure of matrix V, the time spent to compute $V \otimes a^+_{\cdot,M_J}$ is $O(\eta)$. Thus, the time complexity for computing matrix $\widehat{U}$ is $O(\eta(T+\eta)+N\eta)$, where T denotes the time spent to generate $a_{\cdot,j}$. Note also that we only need the storage of $a^+_{\cdot,M_J}$ ($O(\eta)$ space complexity) for computing $\widehat{u}_{\cdot,J}$. Hence, only a part of data are needed at each step of the algorithm. Parameters of procedure **Free** clearly indicate which data are set free in memory at each step $j = M_J, \ldots, m_J$. Thus, the space complexity of the whole algorithm is only $O(\eta)$, which means that it is linear with the number of elements of the state index set S.

The generic function **Coherency** provides another upper bound $\widehat{U}$ on the aggregated dynamics by using one of the Formulae (35) and (41). The specific computation of entries of $\widehat{U}$ is carried out by function **Compute**.

Coherency(A, V, C)

For $J = 1$ to N
  Begin
    For $j = m_J$ to M_J (* loop UP *)
      Begin
      Generate($a_{\cdot,j}$)
        If $j = 1$ then $a^{+}_{\cdot,j} := a_{\cdot,j}$
        else $a^{+}_{\cdot,j}$:= **UpOpt**($a_{\cdot,j}$)
      End
    $\widehat{u}_{\cdot,J}$:= **Compute**($A^{+\cdot,J}, C$)
    Free($a_{\cdot,j}, j = m_J, \ldots, M_J$)
    Free($a^{+}_{\cdot,j}, j = m_J, \ldots, M_J - 1$)
  End
Return $\widehat{U}$

Construction of an upper bound $\widehat{U}$ based on the C-coherency

We list properties that are explicitly used for the computation of the entries of $\widehat{U}$. Since $\mathbb{k}^{I,I}_{i,\cdot}$ is the ith row of matrix $\mathbb{K}_{\eta_I}$, we have for $I, J = 1, \ldots N$:

$$i = 1, \ldots, \eta_I \qquad \mathbb{k}^{I,I}_{i,\cdot} \otimes c^I = c^I_i \oplus \mathbb{k}^{I,I}_{i+1,\cdot} \otimes c^I \tag{56a}$$

$$i = 1, \ldots, \eta_I \qquad \mathbb{k}^{I,I}_{i,\cdot} \otimes A^{I,J} = A^{I,J}_{i,\cdot} \oplus \mathbb{k}^{I,I}_{i+1,\cdot} \otimes A^{I,J} \tag{56b}$$

$$\mathbb{1}^{\top}_{\eta_I} \otimes A^{I,J} = \mathbb{k}^{I,I}_{1,\cdot} \otimes A^{I,J}. \tag{56c}$$

We also have:

$$\bigoplus_{K=I}^{N} \mathbb{1}^{\top}_{\eta_K} \otimes A^{K,J} = \mathbb{1}^{\top}_{\eta_I} \otimes A^{I,J} \oplus \bigoplus_{K=I+1}^{N} \mathbb{1}^{\top}_{\eta_K} \otimes A^{K,J} \tag{56d}$$

We present now the two versions of function **Compute**.

Compute($A^{\cdot,J}, C$)
$\beta := \mathbb{0}_{\eta_J}$
For $I = N$ to 1
  Begin
    $\alpha := \mathbb{0}$; $\gamma := \mathbb{0}_{\eta_J}$; $u = \mathbb{0}$
    For $i = \eta_I$ to 1
      Begin
        $\alpha := \alpha \oplus c^I_i$ — from Relation (56a)
        $\gamma^{\top} := \gamma^{\top} \oplus a^{I,J}_{i,\cdot}$ — from Relation (56b)

$x := (\gamma^\top \oplus \beta^\top) \otimes c^J$
$u := u \oplus (x - \alpha)$ from Relation (35)
End
$\widehat{u}_{I,J} := u$
$\beta := \beta \oplus \gamma$ from Relation (56c)
End
Return $\widehat{u}_{\cdot,J}$

Computation of $\widehat{u}_{\cdot,J}$ from (35)

Compute$(A^{\cdot,J}, C)$
$\beta := \mathbb{0}_{\eta_J}$; $\Sigma_u := \mathbb{0}$
For $I = N$ to 1
Begin
$\alpha := \mathbb{0}$; $\gamma := \mathbb{0}_{\eta_J}$; $u = \mathbb{0}$
For $i = \eta_I$ to 1
Begin
$\alpha := \alpha \oplus c_i^I$ from Relation (56a)
$\gamma^\top := \gamma^\top \oplus a_{i,\cdot}^{I,J}$ from Relation (56b)
$x := (\gamma^\top \oplus \beta^\top) \otimes c^J$
if ($\Sigma_u < x$) then $u := u \oplus (x - \alpha)$ from relation (41)
End
$\widehat{u}_{I,J} := u$
$\Sigma_u := \Sigma_u \oplus \widehat{u}_{I,J}$
$\beta := \beta \oplus \gamma$ from Relation (56c)
End
Return $\widehat{u}_{\cdot,J}$

Computation of $\widehat{u}_{\cdot,J}$ from (41)

Function **Coherency** has with a $O(\eta_J \eta)$-time complexity, whichever the version of function **Compute** that we use. It requires the storage of vectors $a_{\cdot,j}, j = m_J, \ldots, M_J$ and $a^+_{\cdot,j}, j = m_J, \ldots, M_J - 1$, i.e. the function has a $O(\eta\eta_J)$-space complexity. The loop UP has an $O(\eta_J(T + \eta))$- time complexity and an $O(\eta)$-space complexity, recalling that T is the time spent to generate $a_{\cdot,j}$. Thus, the time complexity of the whole algorithm is $O((T + \eta)\eta + \eta^2)$ and its space complexity is $O(\eta \max_{J=1,\ldots N} \eta_J)$.

7 Conclusion

In this paper, we define a new preorder $\leq_{\mathbb{K}}$ for comparing the state vectors of max-plus linear systems. Then, we are interested in bounding the state vectors of lumped max-plus systems with respect to $\leq_{\mathbb{K}}$. The originality of the proposed methodology consists in combining bounds on the state vectors of the linear system and lumpability conditions to have the linear feature for the lumped system. We emphasize that all results are explicit. Hence, we develop algorithms. Their complexity shows that they can be efficient for analyzing large max-plus linear systems. Further investigations will concern the assessment of the quality of bounds. Clearly, the quality should depend on the underlying lumpability criterion and on the "distance" of the matrix governing the dynamics of the initial system from a monotone matrix. Finally, it can be intended to generalize our approach to more general algebraic structures.

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